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Differential equations with definite and practical applications in the electrical field

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DIFFERENTIAL EQUATIONS WITH DEFINITE AND PRACTICAL APPLICATIONS IN THE ELECTRICAL FIELD

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PREFACE

"Differential Equations with Definite and Practical Applications in the Electrical Field"

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Respectfully,

S. Nelson Butler, Jr.

Definitions and Symbols

Differential equations are those in which relations between the changes of variables are found rather than between the variables themselves.

E.M.F. is the influence which can produce an electric current against opposition.

Resistance is the opposition offered to a flow of current.

Inductance is the property of a circuit by virtue of which an opposition is offered to a change of current in a circuit.

Capacitance is a measure of the ease with which a charge may be placed on a body and it is equal to the ratio of the charge to the potential of the charge.

E.M.F. = Electromotive Force

R = Resistance

C = Capacitance

L = Inductance

w = Angular Velocity = $2\pi n$ (n = No. of revolutions)

i = instantaneous current

E = Maximum voltage

x = Reactance

Z = Impedance

α = Time angle (in closing switch)

θ = Phase angle

q = charge

e = naperian base of logarithms

All letters of the alphabet used and not explained are "constants".

INTRODUCTION

OBJECT:

The object of this thesis is to show the applications of differential equations in resistive, inductive and capacitive circuits in the electrical field.

SCOPE:

It has not been the purpose of the writer to present a thesis which will set up a new theory in mathematics and thus add to the general knowledge of all. It would be impossible to do so for two reasons: First, such a piece of work would require much more extensive study in mathematics than the writer has had in addition to an expenditure of a large amount of time. Second, with the limited courses that are available in mathematics, it would be unwise to attempt something beyond the available facilities with the writer lacking knowledge of some of the higher mathematics.

However, it has been the purpose of the writer to present a thesis which will show certain definite and practical applications of differential equations in the electrical field where the solution of particular equations is dependent upon their use. In some instances, two solutions are offered to the same problem to show the flexibility of the differential equation methods. For some of the problems, values for the various quantities have been assumed and their results plotted to give a graphic picture of conditions from time to time. It is hoped that the work and the material developed and compiled in this thesis shows the understanding and grasp that the writer has on the subject and that it represents work characteristic and capable of a graduate student.

TEXT

In the electrical field, a very wide use of vacuum tubes has occurred in the last few years. The "tube circuits" are made up in part of quantities known as resistance, inductance and capacitance. The action of these tubes depends largely upon these quantities and the solving of the equations involving these quantities can only be done by use of the differential equations.

It has been the purpose of this thesis to show the use of differential equations and this particular field demonstrates their worth and practicability. The solution of the problems appears in a text book style, and one who possesses knowledge of the calculus and some electrical theory can readily follow the various steps and processes. Some of the solutions show the simplicity and ease with which the problems are solved and others show the complex forms of solutions. Problem No. 10 is a second order differential equation involving the three quantities, resistance, inductance and capacitance. In this problem, definite assumed values for these quantities have been substituted for their respective symbols and the problem solved just as it would be met in practice.

Problem No. I.

Equation for a circuit containing inductance and resistance is: $e = Ri + L\frac{di}{dt}$

Solution A.

The equation is of a linear form, and can be written as:

(1) $\frac{e}{L} = \frac{di}{dt} + \frac{R}{L}i$

which is similar to : $\frac{dy}{dx} + Py = Q$

where $P = \frac{R}{L}$, $Q = \frac{e}{L}$, $Y = i$ and $t = x$ in this case

Multiplying equation (1) by the integrating factor $\epsilon^{\int P\,dx}$*, we have

$$\epsilon^{\int P\,dx} Q = \epsilon^{\int P\,dx}\frac{dy}{dx} + y\epsilon^{\int P\,dx} P$$

The right side of the equation is the derivative of $Y\epsilon^{\int P\,dx}$

Therefore $\int \epsilon^{P\,dx} Q\,dx + A = Y\epsilon^{\int P\,dx}$

or (2) $y = \epsilon^{-\int P\,dx}\left[\int \epsilon^{\int P\,dx} Q\,dx + A\right]$ (General Equation)

Now returning to equation (1), the solution for the current equation is derived by using equation (2), remembering that $\frac{e}{L} = Q = 0$, $P = \frac{R}{L}$, $y = i$, and $x = t$.

$$\int P\,dt = \int \frac{R}{L}\,dt = \frac{R}{L}t$$

$$i = \epsilon^{-\frac{R}{L}t}\left[\int \epsilon^{\frac{R}{L}t} .0.\,dt + A\right]$$

$$i = \epsilon^{-\frac{R}{L}t}.\left[0 + A\right]$$

(3) $i = A\epsilon^{-\frac{R}{L}t}$

To find A, we assume that the current has attained a value I when the electromotive force is removed. Then when $t = 0$, $i = I_o$

$$i = I_o = A\epsilon^{-\frac{R}{L}\times 0}$$

$$\therefore A = I_o = \frac{E}{R}$$

(A) $i = \frac{E}{R}\epsilon^{-\frac{R}{L}t}$

Solution B.

We start with the assumption that we know the expression for the current is of the form: $i = A\epsilon^{mt}$

Substituting this value of i in the original equation and make $E = 0$

$$0 = RA\epsilon^{mt} + LmA\epsilon^{mt}$$

$$0 = R + L\,m$$

$$m = -\frac{R}{L}$$

$$i = A\epsilon^{-\frac{R}{L}t}$$

Now with the same assumptions as in solution A,

$$i = I_o = A\epsilon^{-\frac{R}{L}\times 0}$$

$$A = I_o = \frac{E}{R}$$

$$i = \frac{E}{R}\epsilon^{-\frac{R}{L}t}$$

* The following shows how the integrating factor $\epsilon^{\int P\,dx}$ is obtained.

Given the linear equation $Q = Py + \frac{dy}{dx}$

Let $Y = u\,V$

Then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ Substituting these values of Y and $\frac{dy}{dx}$ in the given equation. $Q = Puv + u\frac{dv}{dx} + v\frac{du}{dx}$

(2) $Q = v\left(Pu + \frac{du}{dx}\right) + u\frac{dv}{dx}$

Put $\left(Pu + \frac{du}{dx}\right) = 0$ and solve

$$Pu + \frac{du}{dx} = 0$$

$$\frac{du}{u} = -P\,dx$$

$$\int\frac{du}{u} = -\int P\,dx$$

$$\ln u = -\int P\,dx$$

$$u = \epsilon^{-\int P\,dx}$$

By substituting this value of u in equation (2), we will again arrive at the general equation developed under Problem I.

$$\epsilon^{-\int P\,dx}\frac{dv}{dx} = Q$$

$$\frac{dv}{dx} = \epsilon^{\int P\,dx}Q$$

$$dv = \epsilon^{\int P\,dx}Q\,dx$$

$$\int dv = \int \epsilon^{\int P\,dx}Q\,dx$$

$$v = \int \epsilon^{\int P\,dx}Q\,dx + A$$

Since $Y = uv$, then $V = \frac{Y}{u}$

$$\text{and } \frac{Y}{u} = \int \epsilon^{\int P\,dx}Q\,dx + A$$

$$\text{or } Y = \epsilon^{-\int P\,dx}\left[\int \epsilon^{\int P\,dx}Q\,dx + A\right] \qquad \text{since } U = \epsilon^{-\int P\,dx}$$

Problem No. 2

Equation for a circuit containing resistance and inductance is:

$$e = Ri + L\frac{di}{dt}$$

Solution A.

E.M.F. = constant = E. The theory and general equation developed under solution A of Problem I is followed.

$$\frac{e}{L} = \frac{R}{L}i + \frac{di}{dt}$$

$$\frac{e}{L} = \frac{E}{L} = \frac{R}{L}\,i + \frac{di}{dt}$$

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}, \quad y = i, \quad x = t$$

$$\int P\,dt = \int\frac{R}{L}\,dt = \frac{R}{L}\times t$$

$$i = \epsilon^{-\frac{R}{L}t}\int\epsilon^{\frac{R}{L}\times t}\cdot\frac{E}{L}\,dt + A$$

$$i = \epsilon^{-\frac{R}{L}t}\,\frac{E}{L}\cdot\frac{L}{R}\int\epsilon^{\frac{R}{L}\times t}\,\frac{R}{L}\,dt + A$$

$$i = \epsilon^{-\frac{R}{L}t}\,\frac{E}{R}\left[\epsilon^{\frac{R}{L}\times t} + A\right]$$

$$i = \frac{E}{R}\epsilon^{0} + A\epsilon^{-\frac{R}{L}\times t}$$

$$i = \frac{E}{R} + A\epsilon^{-\frac{R}{L}\times t}$$

To find A, get i at some time t
If i = o, t = o

$$0 = \frac{E}{R} + A\epsilon^{-\frac{R}{L}\times 0}$$

$$A = -\frac{E}{R}$$

$$i = \frac{E}{R} - \frac{E}{R}\epsilon^{-\frac{R}{L}\times t}$$

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$$

Solution B.

We start with the assumption that we know the expression for the current is of the form $i = A\epsilon^{mt}$

Substituting this value in the original equation and make $e = E$

$$e = E = RA\epsilon^{mt} + LAm\epsilon^{mt}$$

To solve this equation, let $E = 0$, and make $t = 0$

$$0 = R + Lm$$

$$m = -\frac{R}{L}$$

The current is made up of two parts, a transient part and a permanent part. That is $i = Y + u$

where $Y =$ transient part

$u =$ permanent part

The permanent part $u = \frac{E}{R}$ since the impressed voltage is a constant and the current must then follow Ohm's law.

The transient part $Y = A\epsilon^{-\frac{R}{L}\times t}$

The total current is $= A\epsilon^{-\frac{R}{L}\times t} + \frac{E}{R}$

To find A, get i at some time t

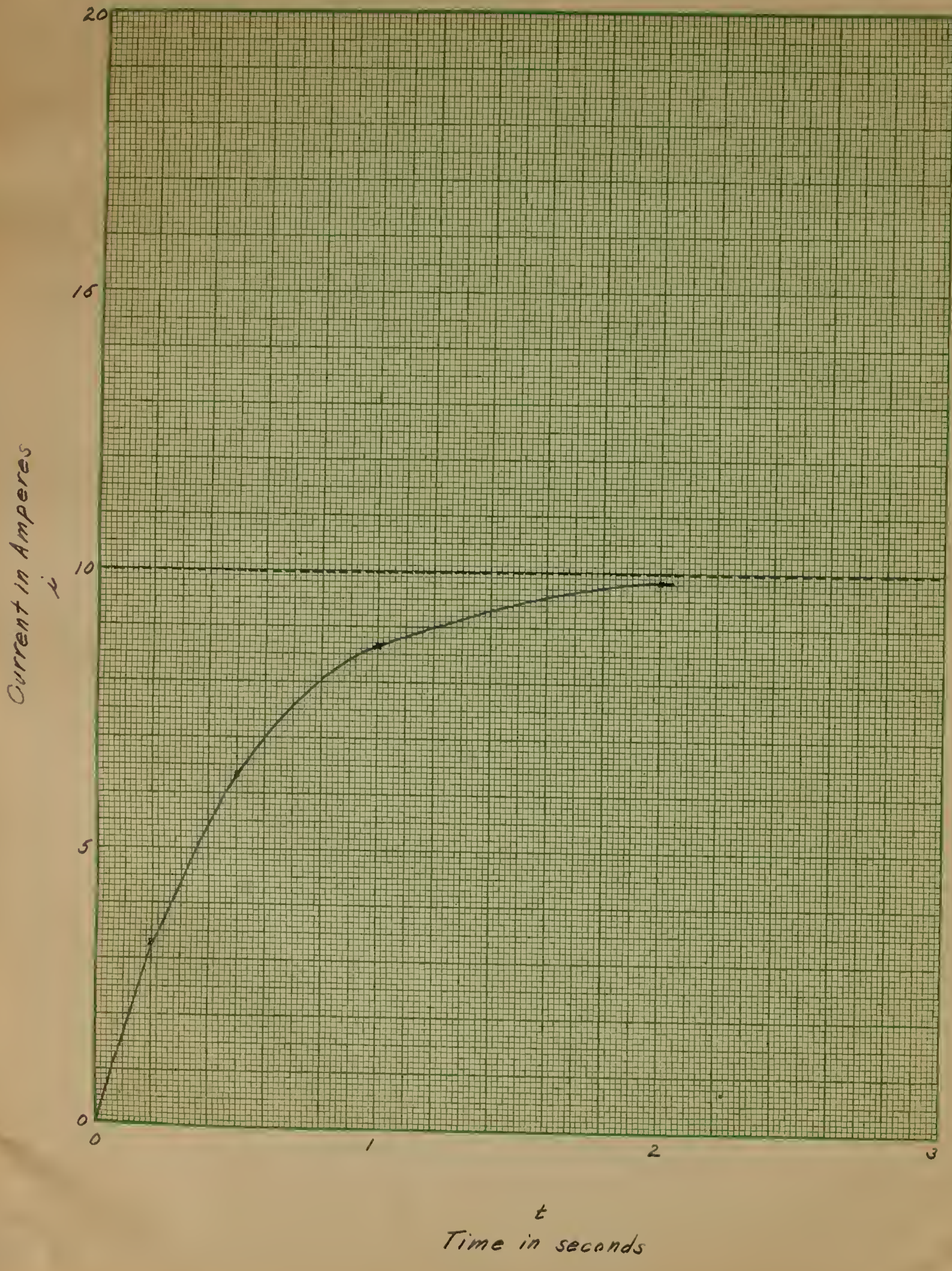
If $i = 0$, $t = 0$

$$i = 0 = A\epsilon^{-\frac{R}{L}\times 0} + \frac{E}{R}$$

$$0 = A + \frac{E}{R} \qquad A = -\frac{E}{R}$$

$$i = -\frac{E}{R}\epsilon^{-\frac{R}{L}\times t} + \frac{E}{R}$$

Graph for Problem No. 2



$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$$

The following numerical values for E, E, L and t have been substituted in the equation for the current developed under Problem No. 2.

E = 100 volts

R = 10 ohms

L = 5 henries

t = 2, 1, o.5, o.2 seconds

Sample substitution: (using t = 2 seconds)

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L} \times t}\right)$$

$$i = \frac{100}{10}\left(1 - \epsilon^{-\frac{10}{5} \times 2}\right)$$

$$i = \frac{100}{10}\left(1 - \epsilon^{-4}\right)$$

$$i = 9.81 \text{ amperes.}$$

Table of results

t (secs.)	i (amps.)
2	9.81
1	8.65
0.5	6.32
0.2	3.30
0	0

Note:

As t increases, $\epsilon^{-\frac{R}{L} \times t}$ approaches 0 and the value of i becomes a constant 10. See graph.

Problem No. 3

Equation for a circuit containing resistance and inductance is:

$$Ri + L\frac{di}{dt}$$

Solution A.

E. M. F. = E sin wt. The theory and general equation developed under solution A of Problem I is followed.

$$e = \frac{E}{L}\sin(wt) \qquad = \frac{R}{L}\, i \,\frac{di}{dt}$$

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}\sin(wt), \qquad Y = i,\ x = t$$

$$\int P\,dt = \int \frac{R}{L}\,dt = \frac{R}{L}\,t$$

(1) $$i = \epsilon^{-\frac{R}{L}t}\int \epsilon^{\frac{R}{L}t}\,\frac{E}{L}\sin(wt)\,dt + A$$

From Pierce's tables of short integrals,(5) we find that

$$\int e^{ax}\sin px\,dx = \frac{e^{ax}(a\sin px - p\cos px)}{a^2+p^2}$$

This form of integral exists in equation (1)

$$i = \epsilon^{-\frac{R}{L}t}\left[\frac{\frac{E}{L}\epsilon^{\frac{R}{L}t}\left[\frac{R}{L}\sin(wt) - w\cos(wt)\right]}{\frac{R^2}{L^2}+w^2} + A\right]$$

$$i = \frac{\frac{E}{L}\frac{R}{L}\epsilon^{0}\sin(wt) - w\frac{E}{L}\epsilon^{0}\cos(wt)}{\frac{R^2}{L^2}+w^2} + A\epsilon^{-\frac{R}{L}\times t}$$

$$i = \frac{E}{L\left(\frac{R^2}{L^2}+w^2\right)}\left(\frac{R}{L}\sin(wt) - w\cos(wt)\right) + A\epsilon^{-\frac{R}{L}\times t}$$

To find A, get i at some time t

If $i = 0$, $t = 0$

$$0 = \frac{E}{L\left(\frac{R^2}{L^2}+w^2\right)}\left(\frac{R}{L}\sin(0^\circ) - w\cos(0^\circ)\right) + A\epsilon^{0}$$

$$i = \frac{E}{L\left(\frac{R^2}{L^2} + w^2\right)} \left(\frac{R}{L} \sin (wt) - w \cos (wt) + \frac{E\,W}{L\left(\frac{R^2}{L^2} + w^2\right)} \epsilon^{-\frac{R}{L} t}\right)$$

Solution B.

We assume that we know the expression for the current is of the form $i = A\epsilon^{mt}$

(1) $$e = R\,i + L\frac{di}{dt}$$

$$e = E \sin (wt + \alpha) = R\,A\epsilon^{mt} + Lm\,A\epsilon^{mt}$$

Let the E. M. F. = 0 at time t = 0

$$0 = R + L\,m$$

$$m = -\frac{R}{L}$$

$$i = A\epsilon^{-\frac{R}{L}t} \quad \text{(transient part)}$$

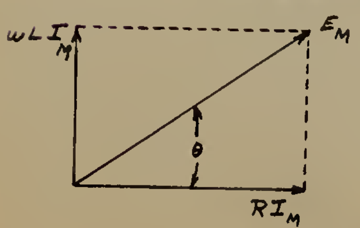
(2) $i = I_M \sin (wt + \alpha + \theta)$ θ = phase angle with respect to E_M

Substituting (2) in (1)

$$E_M \sin (wt + \alpha) = R\,I_M \sin (wt + \alpha + \theta) + w\,L\,I_M \cos (wt + \alpha + \theta)$$

Since a cosine function of time leads a sine function of time, we may write

$$E_M \sin (wt + \alpha) = R\,I_M \sin (wt + \alpha + \theta) + W\,L\,I_M \sin (wt + \alpha + \theta + 90^\circ)$$



$$\theta = -\tan^{-1} \frac{w\,L\,\cancel{I}}{R\,\cancel{I}}$$ θ is negative because voltage lags.

$$\theta = -\tan^{-1} \frac{wL}{R}$$

$$E_M = \sqrt{\overline{R\,I_M}^2 + \overline{w\,L\,I_M}^2}$$

$$= I_M \sqrt{R^2 + \overline{wL}^2}$$

$$I_M = \frac{E_M}{\sqrt{R^2 + w^2 L^2}} = \frac{E_M}{\sqrt{R^2 + x^2}} = \frac{E_M}{z}$$

Total current $I = A\epsilon^{-\frac{R}{L}t} + \frac{E_M}{z}\sin(wt + \alpha + \theta)$

To find A, get i at some time t   If $t = 0$, $i = 0$

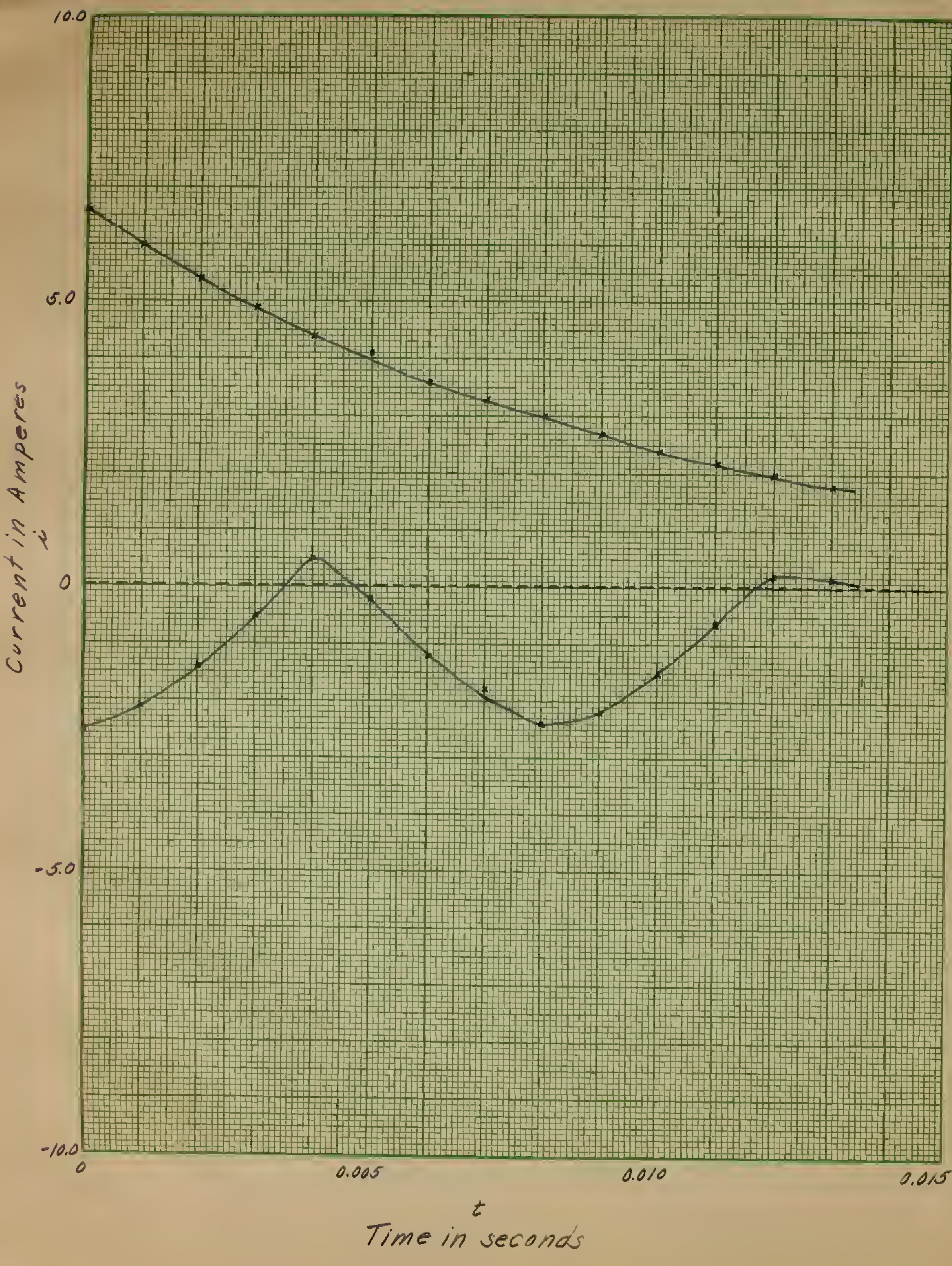
$$0 = A\epsilon^0 + \frac{E_M}{z}\sin(\alpha + \theta)$$

$$A = \frac{E_M}{z}\sin(\alpha + \theta)$$

$$i = \frac{E_M}{z}\sin(\alpha + \theta)\epsilon^{-\frac{R}{L}t} + \frac{E_M}{z}\sin(w\,t + \alpha + \theta)$$

This final equation (i) is a variant form of the final equation for (i) under solution A.

Graph for Problem No. 3



The following numerical values for E, R, L, and t have been substituted in the equation for the current developed under Problem No.3.

E=I000 volts
R=I0 ohms
L=0.I henry
t=0.00I, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009,

0.0I0, 0.0I[illegible], 0.0I3 seconds.

Sample substitution: (using t=0.00I seconds)

$$i = \frac{E}{L\left(\frac{R^2}{L^2}+w^2\right)}\left(\frac{R}{L}\sin wt - w\cos wt\right) + \frac{E}{L\left(\frac{R^2}{L^2}+w^2\right)}\epsilon^{-\frac{R}{L}t}$$

$$i = \frac{1000}{0.1\left(\frac{10^2}{0.1^2}+377^2\right)}\left(\frac{10}{0.1}\sin 377\times 0.001 - 377\cos 377\times 0.001\right) + \frac{1000}{0.1\left(\frac{10^2}{0.1^2}+377^2\right)}\epsilon^{-\frac{10}{0.1}\times 0.001}$$

$$i = 0.66\,(37.1 - 350) + 0.66\times 0.905$$

$$i = -20.7 + 0.060$$

Table of results

t (secs.)	i (amps.)
0.000	-[illegible]
0.00I	-2[illegible].7+0.[illegible]
0.002	-I[illegible].7+0.0[illegible]
0.003	-6.7 +0.[illegible]
0.004	+ 5.I+0.04[illegible]
0.005	-[illegible]+[illegible].04I
0.006	-II.6+0.036
0.007	-I7.[illegible]+0.[illegible]
0.008	-[illegible].[illegible]+0.03[illegible]
0.009	-[illegible].4+0.027
0.0I0	-I5.[illegible]+[illegible].02[illegible]
0.0II	- 6.[illegible] +0.02[illegible]
0.0I2	+I.I[illegible]+0.02[illegible]
0.0I3	+I.3 +0.0I[illegible]

Note: The permanent part and the transient part of the current have been plotted separately. In order to have the graph appear well and be readable, the values of the permanent part have been divided by ten and the values of the transient part have been multiplied by I00.

Problem No. 4

Equation for a circuit containing resistance and inductance is:

$$e = R\,i + L\,\frac{di}{dt}$$

Solution:

E. M. F. $= E_1 \sin(wt) + E_2 \sin(bwt + \theta)$ The theory and general equation developed in solution A of Problem I is followed.

$$e = \frac{E_1}{L}\sin(wt) + \frac{E_2}{L}\sin(bwt + \theta) = \frac{R}{L}\,i + \frac{di}{dt}$$

$$P = \frac{R}{L}, \quad Q = \frac{E_1}{L}\sin(wt) + \frac{E_2}{L}\sin(bwt + \theta), \qquad y = i, \quad x = t$$

$$\int P\,dt = \int \frac{R}{L}\,dt = \frac{R}{L}\,t$$

$$i = \epsilon^{-\frac{R}{L}t} \int \epsilon^{\frac{R}{L}t} \left[\frac{E_1}{L}\sin(wt) + \frac{E_2}{L}\sin(bwt + \theta)\right] dt + A$$

$$(1) \quad i = \epsilon^{-\frac{R}{L}t} \int \epsilon^{\frac{R}{L}t}\, \frac{E_1}{L}\sin(wt)\,dt + \epsilon^{-\frac{R}{L}t} \int \epsilon^{\frac{R}{L}t}\, \frac{E_2}{L}\sin(bwt + \theta)\,dt + A\,\epsilon^{-\frac{R}{L}t}$$

From Pierce's tables of integral, we find that

$$\int \epsilon^{ax} \sin p\,x\,dx = \frac{\epsilon^{ax}(a \sin p\,x - p \cos p\,x)}{a^2 + p^2}$$

This form of integral exists in equation (1)

$$i = \epsilon^{-\frac{R}{L}t}\left[\frac{\epsilon^{\frac{R}{L}t}\, E_1\left(\frac{R}{L}\sin(wt) - w\cos(wt)\right)}{L\left(\frac{R^2}{L^2} + w^2\right)}\right] + \epsilon^{-\frac{R}{L}t}\left[\frac{\epsilon^{\frac{R}{L}t}\, E_2\left(\frac{R}{L}\sin(bwt+\theta) - w\cos(bwt+\theta)\right)}{L\left(\frac{R^2}{L^2} + w^2\right)}\right] + A\,\epsilon^{-\frac{R}{L}t}$$

$$i = \frac{E_1}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[\frac{R}{L}\epsilon^{\circ}\sin(wt) - w\cos(wt)\right] + \frac{E_2}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[\frac{R}{L}\epsilon^{\circ}\sin(bwt+\theta)\right.$$

$$\left. - w\cos(bwt+\theta)\right] + A\epsilon^{-\frac{R}{L}t}$$

To find A, get i at some time t.

If $t = 0$, $i = 0$

$$0 = \frac{E_1}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[\frac{R}{L}\sin 0^{\circ} - w\cos 0\right] + \frac{E_2}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[\frac{R}{L}\sin\theta - \cos(0^{\circ}+\theta)\right] + A\epsilon^{\circ}$$

$$0 = \frac{E_1 w}{L\left(\frac{R^2}{L^2}+w^2\right)} + \frac{E_2 w}{L\left(\frac{R^2}{L^2}+w^2\right)} + \frac{E_2 w\cos\theta}{L\left(\frac{R^2}{L^2}+w^2\right)} + A$$

$$A = \frac{E_1 w}{L\left(\frac{R^2}{L^2}+w^2\right)} + \frac{E\,w}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[1 - \cos\theta\right]$$

$$\therefore i = \frac{E_1}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[\frac{R}{L}\sin(wt) - w\cos(wt)\right] + \frac{E_2}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[\frac{R}{L}\sin(bwt+\theta) -\right.$$

$$\left. w\cos(bwt+\theta)\right] + \left[\frac{E_1 w}{L\left(\frac{R^2}{L^2}+w^2\right)} + \frac{E_2 w}{L\left(\frac{R^2}{L^2}+w^2\right)}\left[1 - \cos\theta\right]\right]\epsilon^{-\frac{R}{L}t}$$

Problem No. 5

Equation for circuit containing resistance and capacitance is:

$$\frac{1}{R}\frac{de}{dt} = \frac{i}{RC} + \frac{di}{dt}$$

Solution:

E. M. F. = constant = E. The theory and general equation developed under Solution A, Problem I is followed.

$P = \frac{1}{RC}, \quad Q = \frac{1}{R} \cdot 0, \quad Y = i, \quad x = t$

Integrating factor here is $e^{-\int p\, dt}$

$$\int P\, dt = \int \frac{dt}{RC} = \frac{t}{RC}$$

$$i = \epsilon^{\frac{-t}{RC}} \int \epsilon^{\frac{t}{RC}} \cdot 0 \cdot dt + A$$

$$i = \epsilon^{\frac{-t}{RC}} \cdot 0 + A\epsilon^{\frac{-t}{RC}}$$

$$\therefore \quad i = A\epsilon^{-\frac{t}{RC}}$$

When $t = o$, $i = I$

$$I_o = \frac{E}{R} = A\,\epsilon^{o}$$

$$A = \frac{E}{R}$$

$$\therefore \quad i = \frac{E}{R}\epsilon^{-\frac{t}{RC}}$$

Problem No. 6

Equation for a circuit containing resistance and capacitance is:

$$\frac{1}{R}\frac{de}{dt} = \frac{i}{Rc} + \frac{di}{dt}$$

Solution:

E. M. F. $=$ E sin (wt) The theory and general equation as developed under Solution A, Problem I is followed.

$e = E \sin (wt)$

$$\frac{1}{R}\frac{de}{dt} = w\,E\cos (wt)$$

$P = \frac{1}{RC}$, $Q = \frac{w\,E}{R}\cos (wt)$, $Y = i$, $x = t$

$$\int P\,dt \quad \int \frac{dt}{RC} = \frac{t}{RC}$$

$$(1)\quad i = \epsilon^{-\frac{t}{RC}} \int \epsilon^{\frac{t}{RC}}\, w\,\frac{E}{R}\cos (wt) + A$$

From Pierce's tables of integrals, we find that

$$\int \epsilon^{ax} \cos p\,x\,dx = \frac{\epsilon^{ax}(a \cos p\,x + P \sin p\,x)}{a^2 + p^2}$$

Equation (1) contains an integral of this form.

$$i = \epsilon^{-\frac{t}{RC}}\left[\frac{\epsilon^{\frac{t}{RC}}\, w\,E\left(\frac{1}{RC}\cos (wt) + w \sin (wt)\right)}{R\left(\frac{1}{R^2C^2} + W^2\right)} + A\right]$$

$$i = \epsilon^{-\frac{t}{RC}}\left[\frac{\epsilon^{\frac{t}{RC}}\frac{wE}{RC}\cos (wt) + \epsilon^{\frac{t}{RC}} w^2 E \sin (wt)}{R\left(\frac{1}{R^2C^2} + w^2\right)} + A\right]$$

$$i = \frac{\epsilon^{0}\frac{wE}{RC}\cos (wt) + \epsilon^{0} w^2 E \sin (wt)}{R\left(\frac{1}{R^2C^2} + w^2\right)} + A\epsilon^{-\frac{t}{RC}}$$

$$i = \frac{w\,E}{R\left(\frac{1}{R^2C^2}+w^2\right)}\left[\frac{1}{RC}\cos(wt) + w\sin(wt)\right] + A\epsilon^{-\frac{t}{RC}}$$

To find A, get <u>i</u> at some time <u>t</u>. If $t = o$, $i = o$

$$0 = \frac{w\,E}{R\left(\frac{1}{R^2C^2}+w^2\right)}\left[\frac{1}{RC}\cos 0^{\circ} + w\sin 0^{\circ}\right] + A\epsilon^{-\frac{t}{RC}}$$

$$0 = \frac{w\,E}{R^2C\left(\frac{1}{R^2C^2}+w^2\right)} + A\,e^{0}$$

$$A = -\frac{C\,w\,E}{1+w^2R^2C^2}$$

$$\therefore\ i = \frac{w\,E}{R\left(\frac{1}{R^2C^2}+w^2\right)}\left[\frac{1}{RC}\cos(wt) + w\sin(wt)\right] - \frac{C\,w\,E}{1+w^2R^2C^2}\epsilon^{-\frac{t}{RC}}$$

Problem No. 7

Equation for a circuit containing resistance and capacity is:

$$R\frac{dq}{dt} + \frac{q}{c} = e$$

Solution A:

E. M. F. = 0 The theory and general equation as developed under Solution A, Problem I is followed.

$E = 0 \quad P = \frac{1}{RC}, \quad Q = E = 0, \quad y = q, \quad x = t$

$$\int P\, d\, t = \int \frac{dt}{RC} = \frac{t}{RC}$$

$$q = \epsilon^{-\frac{t}{RC}} \int \epsilon^{\frac{t}{RC}} \cdot 0 \cdot dt + A$$

$$q = \epsilon^{-\frac{t}{RC}} \cdot 0 + A\epsilon^{-\frac{t}{RC}}$$

$$q = A\epsilon^{-\frac{t}{RC}}$$

To find A, we make t = o by removing the electromotive force and at the instant of short circuiting, the charge q = C E, the value which the charged reached before short circuiting.

$$q = C\ E = A\epsilon^{-\frac{o}{RC}}$$

$$C\ E = A$$

$$q = C\ E\ \epsilon^{-\frac{t}{RC}}$$

To find the equation for the current, differentiate q with respect to t

$$\frac{dq}{dt} = i = \frac{CE}{RC}\epsilon^{-\frac{t}{RC}}$$

$$i = \frac{E}{R}\epsilon^{-\frac{t}{RC}}$$

Solution B.

We start with the assumption that we know the expression for the charge is of the form $q = A\epsilon^{mt}$, then $\frac{dq}{dt} = m A\epsilon^{mt}$

Substituting these values of q and $\frac{dq}{dt}$ in the given equation, we get

$$R\, m\, A\epsilon^{mt} + \frac{A\epsilon^{mt}}{C} = e$$

and since $e = o$,

$$R\, m' + \frac{m^{\circ}}{C} = 0$$

$$m = -\frac{1}{RC}$$

$$\therefore\ q = A\epsilon^{-\frac{t}{RC}}$$

To find A, we make $t = o$ by short circuiting the electromotive force. The value at the charge q of the instant of short circuiting is CE.

$$\therefore\ q = C\,E = A\epsilon^{-\frac{o}{RC}}$$

$$A = C\,E$$

$$\therefore\ q = C\,E\,\epsilon^{-\frac{t}{RC}}$$

To find the expression for the current, differentiate q with respect to t.

$$\frac{dq}{dt} = i = \frac{CE}{RC}\epsilon^{-\frac{t}{RC}}$$

$$i = \frac{E}{R}\epsilon^{-\frac{t}{RC}}$$

Problem No. 8

Equation of a circuit containing resistance and capacitance is: $R\frac{dq}{dt} + \frac{q}{c} = e$

Solution

E. M. F. = Constant = E. To solve put E = o and let $q = A\epsilon^{mt}$.

$$\frac{dq}{dt} = m A \epsilon^{mt}$$

$$e = 0 = R m A\epsilon^{mt} + \frac{A\epsilon^{mt}}{c}$$

$$0 = R m' + \frac{m^{\circ}}{c}$$

$$m = -\frac{1}{RC}$$

The total equation is the transient part plus the steady part

$$q = A\epsilon^{-\frac{t}{RC}} + CE$$

To find A, the constant of integration, let t = o, then q = o

$$0 = A\epsilon^{0} + CE$$

$$A = - CE$$

$$q = - CE\epsilon^{-\frac{t}{RC}} + CE$$

$$q = CE(1 - \epsilon^{-\frac{t}{RC}})$$

To find the expression for current, differentiate q with respect to t.

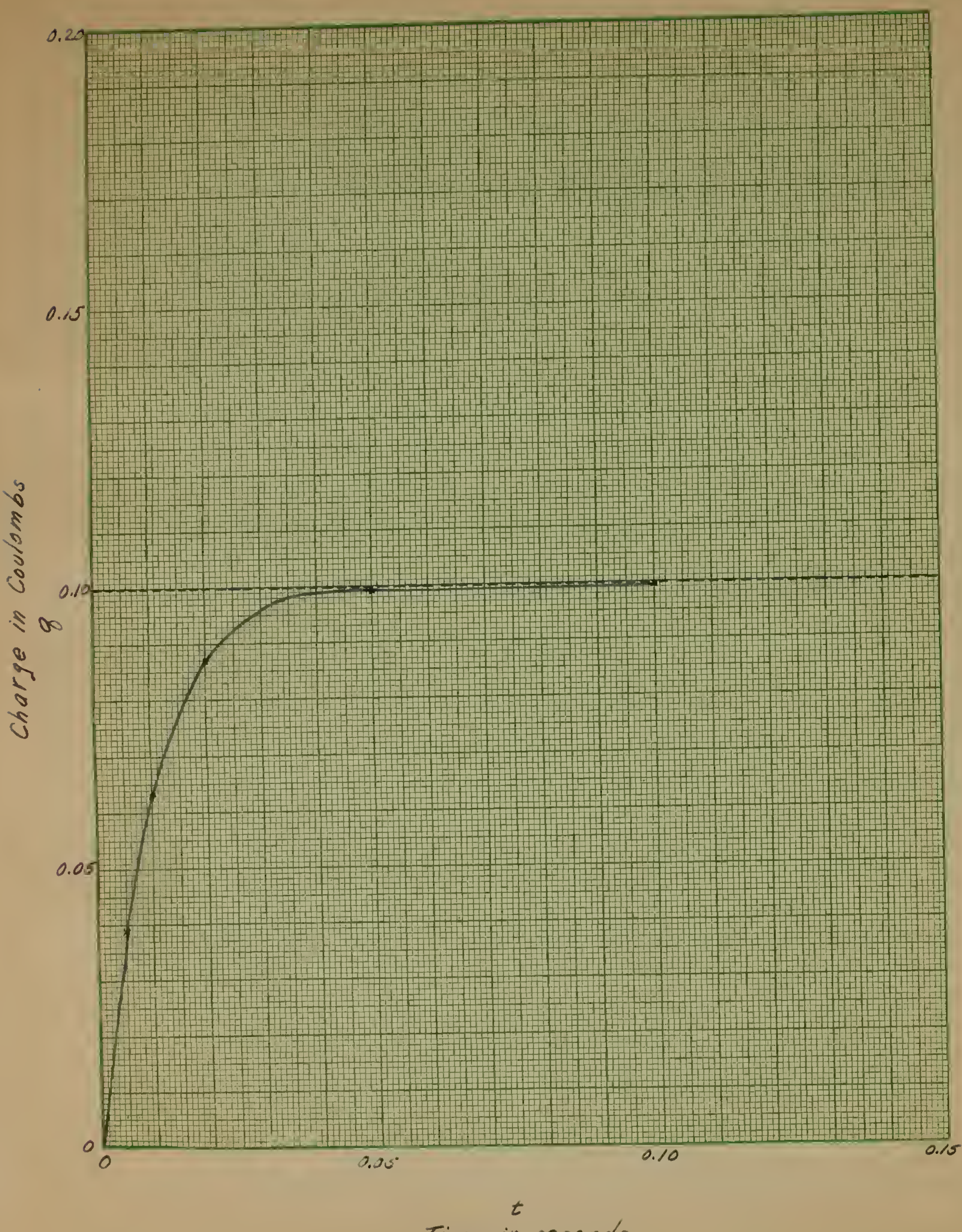
$$\frac{dq}{dt} = 0 - CE.(-\frac{1}{RC})\epsilon^{-\frac{t}{RC}}$$

$$\frac{dq}{dt} = i = \frac{CE}{RC}\epsilon^{-\frac{t}{RC}}$$

$$i = \frac{E}{R}\epsilon^{-\frac{t}{RC}}$$

The following values for R, E, C and t have been substituted in

Graph for Problem No. 8



the equation for the charge developed under Problem No. 8.

$E = 1000$ volts

$R = 100$ ohms

$C = 10^{-4}$ farads

$t = 0.1, 0.05, 0.02, 0.01, 0.005$ seconds.

Sample substitution (Using $t = 0.1$ second)

$$q = CE(1 - \epsilon^{-\frac{t}{cr}})$$

$$= 10^{-4} \times 1000 \left(1 - \epsilon^{-\frac{0.1}{10^{-4} \times 100}}\right)$$

$$= 10^{-1} (1 - \epsilon^{-10})$$

$$= 10^{-1}$$

$$= 0.1$$

Table of Results

t(secs.)	q(coulombs)
0.1	0.0999
0.05	0.0993
0.02	0.087
0.01	0.063
0.005	0.039
0	0

Problem No. 6

Equation of a circuit containing resistance and capacitance is: $R\frac{dq}{dt}+\frac{q}{c}=e$

Solution A:

E. M. F. $=$ E sin (wt)

$$E \sin (wt) = R\frac{dq}{dt}+\frac{q}{c}$$

$$\frac{E}{R} \sin (wt) = \frac{dq}{dt} + \frac{q}{RC}$$

$$P = \frac{1}{RC}, \quad Q = \frac{E}{R} \sin (wt) \qquad y = q, \quad x = t$$

Here the integrating factor is $\epsilon^{-\int P\,dt} = \epsilon^{\int\frac{dt}{RC}} = \epsilon^{-\frac{t}{RC}}$

$$q = \epsilon^{-\frac{t}{RC}}\left[\int \epsilon^{\frac{t}{RC}} \frac{E}{R} \sin (wt)\, dt + A\right]$$

From Pierce's Tables of Integrals, we find that:

$$\int \epsilon^{ax} \sin p\,x = \frac{\epsilon^{ax}(a \sin p\,x - p \cos p\,x)}{a^2+p^2}$$

$$q = \epsilon^{\frac{t}{RC}} \frac{E}{R}\left[\epsilon^{\frac{t}{RC}} \frac{\left(\frac{1}{RC} \sin (wt) - w \cos (wt)\right)}{\frac{1}{R^2C^2} + w^2} + A\right]$$

$$q = \frac{\frac{\epsilon^{o} E}{R\ C} \sin (wt) - \epsilon^{o}\frac{Ew}{R} \cos (wt)}{\frac{1+w^2R^2C^2}{R^2C^2}} + A\epsilon^{-\frac{t}{RC}}$$

$$q = \frac{CE}{1+w^2R^2C^2}\left(\sin (wt) - R\ C\ w \cos (wt)\right) + A\epsilon^{-\frac{t}{RC}}$$

To find A, get $\underline{q}$ at some time $\underline{t}$. If $t = o$, $q = o$

$$0 = \frac{CE}{1+w^2R^2C^2}(\sin 0^\circ - R\ C\ w \cos 0^\circ) + A\ \epsilon^{o}$$

$$A = -\frac{R C^2 E w}{1+w^2R^2C^2}$$

$$\therefore q = \frac{CE}{1+w^2R^2C^2}\left(\sin(wt) - R C w \cos(wt)\right) - \frac{C^2 R E w}{1+w^2R^2C^2}\epsilon^{-\frac{t}{RC}}$$

Equation of a circuit containing resistance and capacitance is: $R\frac{dq}{dt}+\frac{q}{c}=e$

Solution B. Conditions are the same as in Problem No. 9, but in this case the time integral (α) is taken into consideration.

$$e = E_M \sin(wt+\alpha)$$

$$E \sin(wt+\alpha) = Ri+\frac{1}{c}\int i\,dt + k \qquad \text{since } i = \frac{dq}{dt}$$

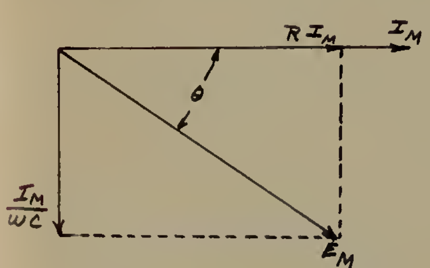
$$i = I_M \sin(wt+\alpha+\theta)$$

$$E \sin(wt+\alpha) = R I_M \sin(wt+\alpha+\theta) + \frac{1}{c}\int I_M \sin(wt+\alpha+\theta)$$

$$= R I_M \sin(wt+\alpha+\theta) + \frac{1}{c}\frac{1}{w} I_M \cos(wt+\alpha+\theta)$$

Since a cosine function of time leads a sine function, we may write

$$E_M \sin(wt+\alpha) = R I_M \sin(wt+\alpha+\theta) - \frac{I_M}{wc}\sin(wt+\alpha+\theta+90^\circ)$$



$$E_M = \sqrt{R I_M^2 \frac{I_M}{wc}^2}$$

$$= I_M\sqrt{R^2+\left(\frac{I}{wc}\right)^2}$$

$$= I_M\sqrt{R^2+x^2}$$

$$= I_M z$$

$$I_M = \frac{E_M}{z}$$

Transient part $= A\epsilon^{-\frac{t}{RC}}$

Permanent part $= I_M \sin(wt+\alpha+\theta) = \frac{E_M}{z}\sin(wt+\alpha+\theta)$

$$\therefore i = A\epsilon^{-\frac{t}{RC}} + \frac{E_M}{z}\sin(wt+\alpha+\theta)$$

To find A, get $\underline{i}$ at some time $\underline{t}$. If $t = 0$, $i = 0$

$$0 = A\epsilon^{0} + \frac{E_M}{Z} \sin(\alpha + \theta)$$

$$A = -\frac{E_M}{Z} \sin(\alpha + \theta)$$

$$\therefore \; i = \frac{E_M}{Z} \sin(wt + \alpha + \theta) - \frac{E_M}{Z} \sin(\alpha + \theta)$$

Problem No. 10

The following data is given to be used in the equation of a circuit containing resistance, inductance and capacitance connected in series.

$R = 5$ ohms

R of the inductive coil $= 5$ ohms

$L = 1$ henry

$C = 10^{-4}$ farads

$E = 50$ volts

Given Equation:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = E$$

(1) $\frac{d^2q}{dt^2} + \frac{5+5}{1}\frac{dq}{dt} + \frac{1}{1X10^{-4}}q = 50$

Let the right hand member $= 0$

(2) $\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 10^4 q = 0$

Substitute for q, e^{mt}

$m^2 e^{mt} + 10m\,e^{mt} + 10^4 = 0$

(3) $m^2 + 10\,m + 10^4 = 0$

Solving for m: $m = \frac{-10 \pm \sqrt{10^2 - 4x10^4}}{2}$

$$m = \frac{-10 \pm \sqrt{100-40,000}}{2}$$

$$m = \frac{-10 \pm \sqrt{-39,900}}{2}$$

$$m = -5 \pm \frac{10}{2}\sqrt{-399}$$

$$m = -5 \pm 5\sqrt{399}\,\sqrt{-1}$$

$$m = -5 \pm \sqrt{9975}\,\sqrt{-1}$$

$$m = -5 \pm 99.9\sqrt{-1}$$

$$\therefore\ e^{mt} = e^{(-5+99.9\sqrt{-1})t}$$

$$e^{m_2 t} = e^{(-5 - 99.9\sqrt{-1})t}$$

$$\therefore \quad q = C_1 e^{(-5 + 99.9\sqrt{-1})t} + C_2 e^{(-5 - 99.9\sqrt{-1})t}$$

$$q = C_1 \left[e^{-5t} e^{99.9\sqrt{-1}\, t} \right] + C_2 \left[e^{-5t} e^{-99.9\sqrt{-1}\, t} \right]$$

$$q = e^{-5t} \left(C_1 e^{99.9\sqrt{-1}\, t} + C_2 e^{-99.9\sqrt{-1}\, t} \right)$$

$$q = e^{-5t} \left[C_1 (\cos bt + \sqrt{-1} \sin bt) + C_2 (\cos bt - \sqrt{-1} \sin bt) \right]$$

$$q = e^{-5t} \left[(C_1 + C_2) \cos bt + (C_1 - C_2)\sqrt{-1} \sin bt \right]$$

(4) $$q = e^{-5t} \left[A \cos bt + B\sqrt{-1} \sin bt \right]$$ (complementary solution of (1)

where $A = C_1 + C_2$ and $B = C_1 - C_2$.

Differentiating (1), we get

(5) $$\frac{d^3 q}{dt^3} + 10 \frac{d^2 q}{dt^2} + 10^4 \frac{dq}{dt} = 0$$

Adding equation (2) to (5), we get

(6) $$\frac{d^3 q}{dt^3} + 9 \frac{d^2 q}{dt^2} + 9990 \frac{dq}{dt} + 10^4 q = 0$$

Let $q = e^{mt}$

(7) $$m^3 e^{mt} + 9 m^2 e^{mt} + 9990\, m\, e^{mt} + 10^4 = 0$$

Using auxillary equation (3), we get $m + 1 = 0$ by dividing (7) by (3)

$\therefore \; m_3 = -1$ where $m_3 =$ third root of auxillary equation

(8) $\therefore \; q = C_3 e^{m_3 t} = C_3 e^{-t}$ (as a particular solution of eq. (1)

Substituting (8) in (1), we get

$$C_3 e^{-t} + 10 C_3 e^{-t} + 10^4 C_3 e^{-t} = 50$$

$$10{,}011\, C_3 e^{-t} = 50$$

$$C_3 = \frac{50}{10{,}011\, e^{-t}}$$

when $t = 0$, $e^{-t} = e^{0} = 1$

$$\therefore C_3 = \frac{50}{10{,}011}$$

$$\therefore C_3 = 0.005$$

Complete solution of original equation is:

(9) $$q = e^{-5t}\left[A \cos bt + B\sqrt{-1} \sin bt\right] + 0.005$$

To solve for A and B, find $\underline{q}$ at some time $\underline{t}$. If $t = o$, $q = o$

$$0 = e^{0}\left[A \cos 0^{\circ} - B\sqrt{-1} \sin 0^{\circ}\right] + 0.005$$

$\therefore A = 0.005$

To solve for B, differentiate (9)

$$\frac{dq}{dt} = i = e^{-5t}\left[- A\, b \sin bt + B\sqrt{-1}\, b \cos bt\right] + \left[A \cos bt + B\sqrt{-1} \sin bt\right] \times e^{-5t} \times 5$$

When $t = o$, $i = o$

$$0 = e^{0}\left[-Ab \sin 0^{\circ} + B\sqrt{-1}\, b \cos 0^{\circ}\right] + \left[A \cos 0^{\circ} + B\sqrt{-1} \sin 0^{\circ}\right] e^{0} \times 5$$

$$0 = B\sqrt{-1}\, b + 5A$$

$$B\sqrt{-1} = -\frac{5A}{b} \qquad \text{(but } b = 99.9\text{)}$$

$$\therefore B\sqrt{-1} = -\frac{5 \times 0.005}{99.9} = -\frac{0.025}{99.9} = -0.00025$$

Therefore the final equation, with all constants known, is

$$q = e^{-5t}\left[-0.005 \cos bt - 0.00025 \sin bt\right] + 0.005$$

(10) $$q = -e^{-5t}\left[0.00025 \sin bt + 0.005 \cos bt\right] + 0.005$$

To find the equation for the current, differentiate eq. (10)

$$i = -e^{-5t}\left[+0.00025\, b \cos bt - 0.005\, b \sin bt\right] + \left[0.00025 \sin bt + 0.005 \cos bt\right] e^{-5t} \times +5$$

$$i = e^{-5t}\left[-0.00025\, b \cos bt + 0.005b \sin bt + 0.00125 \sin bt + 0.025 \cos bt\right]$$

Since $b = 99.9$

$$i = \epsilon^{-5t}\left[-0.025\cos bt+0.4995\sin bt+0.00125\sin bt+0.025\cos bt\right]$$

$$i = 0.501\,\epsilon^{-5t}\sin 99.9t$$

CONCLUSION

In solving the fore-going problems, double solutions have been offered in some cases. All of the problems could be solved by at least two methods. If one has a knowledge of the electrical field, the solutions can be simplified somewhat as demonstrated in several of the preceding problems. Simplifications consist in letting the quantity to be solved for, equal to $A\epsilon^{mt}$; then substituting it and its derivatives in the original equation, and solving the resulting equation. This would naturally be the course for the one using electrical theory to follow. However, the mathematician would attack the problem strictly from a mathematical view point, but both would reach the same conclusion.

APPENDIX

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